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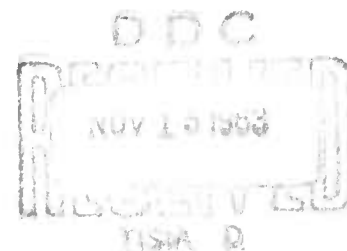
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# ADAPTIVE CONTROL VIA QUASILINEARIZATION AND DIFFERENTIAL APPROXIMATION

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*The* **RAND** *Corporation*  
SANTA MONICA • CALIFORNIA

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PREFACE

An adaptive controller is one which has the capability to learn from experience and in so doing improves the quality of the control it exerts over a plant. Such a controller adapts itself to circumstances as it finds them. The potential military and commercial applications of such devices are impressive.

The aim of this Memorandum is to show that the basic system identification and state determination problems can be viewed, mathematically, as nonlinear multipoint boundary value problems. They can be resolved, computationally, via use of quasilinearization and differential approximation. The net result is that we now possess straightforward and effective procedures for designing wide classes of adaptive controllers. This report may be of interest to control engineers and numerical analysts.

SUMMARY

Suppose that a system undergoes a process described by a system of differential equations. The equations contain some unknown parameters and not all the initial conditions are known. Observations are made on some of the state variables during the course of the process. We wish to determine the system parameters and initial conditions which lead to best agreement with the observations.

Inverse problems of the type sketched, which are important throughout mathematical physics and engineering, are cast in mathematical form as nonlinear multipoint boundary value problems. Then computational solutions via quasilinearization and differential approximation are suggested.

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## I. INTRODUCTION

What distinguishes an adaptive control system from conventional control systems has been a matter of great discussion and controversy. It has even been pointed out that any control system is adaptive if one chooses to define certain functions of a control system in a suitable manner.<sup>(1)</sup>

A commonly accepted definition of an adaptive system is the following: An adaptive control system is one in which certain parameters of the system are adjusted to counteract degradations in performance brought about by changes in the system's environments. Even this definition is rather vague. With this definition the same control system can be adaptive or non-adaptive, depending on its block diagram representation. For examples of this situation see Ref. 2. In this Memorandum the operational definitions given in the reference will be used to distinguish adaptive from non-adaptive control systems. These definitions will be outlined in a later section. The plant refers to the object to be controlled, and system refers to the combination of the plant and controller.

A control problem which can often be solved by conventional techniques is the following: Given a plant whose characteristics cannot be changed by the designer, and given some performance specifications and constraints, find a satisfactory controller to actuate the plant. If the input to the controller is sensitive only to the system input, the resulting system is called open loop. This will yield a satisfactory system only if the plant and its environment and the expected inputs are precisely known beforehand. In general, this will not be

the case. When open loop control is not possible, the controller should also operate on the plant output and the system is then called closed loop. For a closed loop system design to be satisfactory, the plant characteristics and their relationship to environmental changes should be known beforehand.

Closed loop controller design may be viewed in the following manner. Before starting the design procedure, the plant characteristics are known, i.e., the plant has been identified. Based on certain performance specifications or performance indices, a controller is designed, i.e., a decision is made concerning the type of controller to be used. Finally, based on this decision, certain signals or parameters of the system must undergo modification to accomplish this result.

Now consider the problem of designing a controller for a plant with incompletely defined dynamical characteristics. If the dynamical characteristics do not change very rapidly, one can still view this problem as a closed loop controller design. The only difference is that the controller will be different for different environmental conditions. In such cases it may be advisable to so organize the over-all system that it performs certain of the functions conventionally exercised by the designer himself during the design phase. That is, the controller may be called on to compute or identify the characteristics of the plant while the system is in normal operation. The controller must then make a decision concerning the way in which the available parameters of the system should be adjusted so as to improve the operation with respect to a defined index of performance.

Finally, certain signals or parameters must undergo modification to accomplish this result.

Consistent with the definition in Ref. 2, a control system which accomplishes these three functions of identification, decision, and modification will be termed an adaptive control system.

A schematic representation of an adaptive control system is shown in Fig. 1.

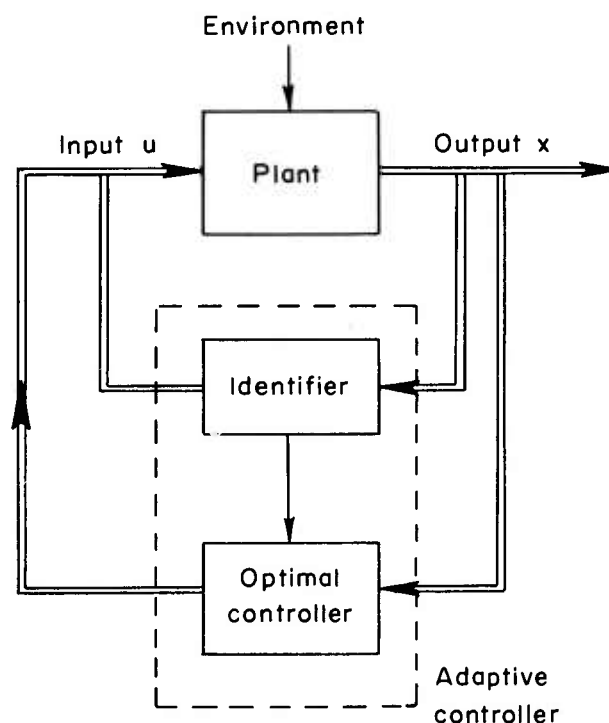


Fig. 1 — Schematic of an adaptive control system

In Fig. 1, double lines indicate vector valued signals. The identification function is performed by the identifier and the decision and modification functions are performed by the optimal controller. The combination of the identifier and optimal controller will be called an adaptive controller.

In this Memorandum the identification and decision problems will be separately considered, and methods for obtaining their solutions proposed. One method of solution depends on converting the identification and decision problems to a problem of solving nonlinear ordinary differential equations with multiple point boundary values specified and solving the latter problem by the method of quasilinearization.<sup>(3)</sup> Another method depends on using differential approximation techniques.<sup>(4)</sup>

For an excellent survey of other methods of solving certain classes of adaptive control problems and the historical development of this area, refer to Gibson.<sup>(5)</sup> See also Ref. 6.

## II. THE IDENTIFICATION PROBLEM

Most of the identification schemes proposed in the control engineering literature involve determination of the impulse response of plants by either subjecting them to specific test signal inputs or using normal operating data. These schemes suffer from the disadvantage that they are useful only for linear time-invariant plants.

The identification schemes proposed here are equally applicable for linear and nonlinear plants. It is assumed that the plants are described by ordinary differential equations whose forms are known except for a finite set of unknown parameters which depend on the environments. The parameters are assumed to be constant over the time required to identify.

The assumption regarding knowledge of the form of the differential equation is not restrictive in practice since this can either be obtained from physical considerations or by letting the identifier go through an initial period of "learning" over which the best form that relates the input-output data may be determined.

Let the dynamical equations of the plant be of the form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, u, \underline{a}) \quad (1)$$

where  $\underline{x}$  is an  $n$ -dimensional vector  $(x_1, \dots, x_n)$ , the state of the system;  $u$  is a scalar, the control function;  $\underline{f}$  is an  $n$ -dimensional vector  $(f_1, \dots, f_n)$ ; and  $\underline{a}$  is a  $k$ -dimensional vector representing the  $k$  parameters of the plant which depend on environmental conditions.

More generally,  $u$  can be an  $m$ -dimensional vector (i.e., the plant is a multi-variable plant) and  $\underline{f}$  can be an explicit function

of time  $t$  (i.e., the plant is a time variable plant). The generalization of the methods proposed below for these cases is straightforward and hence only the special case of plants represented by Eq. (1) will be considered here.

The two schemes for solving the identification problem are the following.

#### A. THE QUASILINEARIZATION METHOD FOR IDENTIFICATION<sup>(7)</sup>

To use this method, the identifier should have the following information. The input to the plant,  $u(t)$ , should be known in the interval  $0 \leq t \leq T$ , where  $T$  is a suitably chosen constant. In addition, observations of the output at certain instants of time should also be known. The observations need only be on the physically available components of the state  $\underline{x}$  of the plant. Let  $\underline{y}$  be a subset of  $\underline{x}$  where the components of  $\underline{y}$  are physically available for measurement. Thus, the known observations are

$$\underline{y}(t_i) = \underline{c}_i \quad (2)$$

$i = 1, 2, \dots, m$  with  $t_m \leq T$  and  $m$  is a suitably chosen number such that Eq. (2) yields  $(n+k)$  conditions.

The parameter vector  $\underline{a}$  is really a constant in the interval  $0 \leq t \leq T$ . However, we shall treat it as a function of time and adjoin to the plant Eq. (1), the equation

$$\dot{\underline{a}} = 0 \quad (3)$$

The question now is the following. What is the solution to the  $(n+k)$  first order Eqs. (1) and (3) subject to the  $(n+k)$  boundary

conditions represented by Eq. (2)? This is a multipoint boundary value problem to be solved computationally using the method of quasi-linearization outlined in Refs. 3, 7, 8, 9, 10.

The linearized equations for the  $(n+1)^{\text{st}}$  approximation are

$$\begin{aligned} \dot{\underline{x}}^{(n+1)} = & \underline{f}(\underline{x}^{(n)}, u, \underline{a}^{(n)}) + \underline{f}_{\underline{x}}^{(n)} (\underline{x}^{(n+1)} - \underline{x}^{(n)}) \\ & + \underline{f}_{\underline{a}}^{(n)} (\underline{a}^{(n+1)} - \underline{a}^{(n)}) \end{aligned} \quad (4)$$

$$\dot{\underline{a}}^{(n+1)} = 0$$

with boundary conditions

$$\underline{y}^{(n+1)}(t_1) = \underline{c}_1 \quad (5)$$

In Eq. (4)

$$\underline{f}_{\underline{x}}^{(n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \begin{cases} \underline{x} = \underline{x}^{(n)} \\ \underline{a} = \underline{a}^{(n)} \end{cases} \quad (6)$$

and  $\underline{f}_{\underline{a}}^{(n)}$  is similarly defined.  $\underline{x}^{(n)}$  is the value of  $\underline{x}$  in the  $n^{\text{th}}$  iteration and  $\underline{a}^{(n)}$  has a similar meaning.

The solution of Eqs. (4) and (5) yields the value of the parameter vector  $\underline{a}$ . Notice that in addition to determining the values of the

unknown parameters the inaccessible states will also be automatically determined by this procedure, which is quadratically convergent.

#### B. THE DIFFERENTIAL-APPROXIMATION METHOD FOR IDENTIFICATION

The current estimate of the plant parameter vector  $\underline{a}$  is the one that makes, from Eq. (1)

$$\dot{\underline{x}} - \underline{f}(\underline{x}, u, \underline{a}) \equiv 0 \quad (7)$$

the identity holding over the identification time. Any other value of the parameter vector will not satisfy Eq. (7). This simple observation makes it evident that the true value of the plant parameters is the solution of

$$\text{Min}_{\underline{a}} \int_0^T < \dot{\underline{x}} - \underline{f}(\underline{x}, u, \underline{a}) , \quad \dot{\underline{x}} - \underline{f}(\underline{x}, u, \underline{a}) > dt \quad (8)$$

where  $T$  is a suitably chosen quantity. In Eq. (8)  $< \cdot >$  denotes the Euclidean inner product.

This is a straightforward minimization problem which yields upon taking partial derivatives of the integral with respect to the components of  $\underline{a}$  and setting them equal to zero

$$\int_0^T \underline{f}'_{\underline{a}} \dot{\underline{x}} dt = \int_0^T \underline{f}'_{\underline{a}} \underline{f} dt \quad (9)$$

In Eq. (9)

$$\underline{f}'_{\underline{a}} = \begin{bmatrix} \frac{\partial f_1}{\partial a_1} & \cdots & \frac{\partial f_n}{\partial a_1} \\ \frac{\partial f_1}{\partial a_n} & \cdots & \frac{\partial f_n}{\partial a_k} \end{bmatrix} \quad (10)$$

Equation (9) represents  $k$  simultaneous equations whose solution yields the plant parameter vector  $\underline{a}$ .

This particular method of identification appears quite promising for on-line applications. However, the disadvantages of this method are that it is necessary to measure all the state variables of the system and in addition, some differentiation operations have to be performed on some of the state variables.

As an example illustrating this method, let the plant to be identified be governed by dynamical equations of the form

$$\frac{d^2x}{dt^2} + a \frac{dx}{dt} + x = K u(t) \quad (11)$$

where  $a$  and  $K$  are the unknown parameters.

Equation (11) represents a system whose block diagram is shown in Fig. 2.

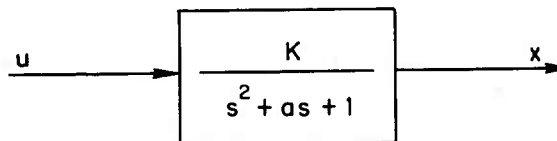


Fig. 2 — Block diagram of example system

In this example, it is seen that the environment affects the gain and time constants of the plant. The true values of  $a$  and  $K$  are the solution of

$$\text{Min}_{a,K} \int_0^T \left\{ \frac{d^2x}{dt^2} + a \frac{dx}{dt} + x - K u(t) \right\}^2 dt \quad (12)$$

The minimization of this expression yields

$$\begin{aligned}
 a \int_0^T \left( \frac{dx}{dt} \right)^2 dt - K \int_0^T \frac{dx}{dt} u(t) dt &= - \int_0^T \left( \frac{d^2x}{dt^2} + x \right) \frac{dx}{dt} dt \\
 a \int_0^T \frac{dx}{dt} u(t) dt - K \int_0^T u^2(t) dt &= - \int_0^T \left( \frac{d^2x}{dt^2} + x \right) u(t) dt
 \end{aligned}
 \tag{13}$$

Notice that the coefficients of  $a$  and  $K$  in the two simultaneous linear Eqs. (13) can be easily generated using multipliers, summers, and integrators if  $x$ ,  $\frac{dx}{dt}$ ,  $\frac{d^2x}{dt^2}$  and  $u(t)$  are available to the identifier.

### III. THE DECISION PROBLEM

Once the plant parameters have been identified, the next problem facing the adaptive controller is the decision problem. A "control function" or a "control law" has to be synthesized to minimize a suitably chosen performance index. Here a control function is defined as a control signal which is only a function of time. A control law is defined as a control signal which is some function of the state of the system. The control law may involve time explicitly.

The decision problem of adaptive control is equivalent to the optimal control problem in automatic control theory. Two methods of solving the decision problem will be outlined with reference to a fairly typical problem in optimal control theory.

Let the identified plant be governed by dynamical equations of the form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, u) \quad (14)$$

The notation in Eq. (14) is the same as in Eq. (1), and the comments made in connection with Eq. (1) are also pertinent to Eq. (14).

Let the plant be in an initial state

$$\underline{x}(0) = \underline{c} \quad (15)$$

Determine  $u(t)$  to minimize a performance index of the form

$$I(u) = \int_0^T g(\underline{x}, u) dt \quad (16)$$

where  $g(\underline{x}, u)$  is a scalar valued function of its arguments.

More generally, certain constraints like bounds on  $u(t)$  may be placed on the minimization problem. However, such constraints may be easily eliminated by suitable transformations. (For example, see Ref. 11.) Hence, only the unconstrained optimization problem mentioned above will be considered to outline the methods for solving the decision problem.

The Euler equations for the optimization problem are Eq. (14) together with

$$\dot{\underline{\lambda}} = -\underline{g}_x - \underline{f}'_x \underline{\lambda} \quad (17)$$

$$\underline{g}_u + \langle \underline{\lambda}, \underline{f}_u \rangle = 0 \quad (18)$$

where

$$\underline{g}_x = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix} \quad (19)$$

$$\underline{f}'_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (20)$$

and

$$\underline{f}_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix} \quad (21)$$

In Eqs. (17) and (18),  $\underline{\lambda}$  is an  $n$ -vector, the Lagrangian multiplier vector.

Equations (17) and (18) have to be solved with the boundary conditions given by Eq. (15) and the free end condition

$$\underline{\lambda}(T) = 0 \quad (22)$$

The Euler equations can be solved by the method of quasilinearization to yield the optimal control function or open loop solution in the form

$$\mu = \psi(t) \quad (23)$$

and the optimal system trajectory in the form

$$\underline{x} = \varphi(t) \quad (24)$$

A solution in this form is unsatisfactory in general and a control law is usually desired. It is not easy to obtain the control law from the control function.

In many practical situations, even if a control law can be synthesized, it will not be a satisfactory solution because of the complexity of the dependence of the control law on the state of the system.

Often, the form of dependence of the control law on the state is known beforehand except for a finite set of parameters. The known form depends on the manipulations that are possible with the available physical equipment.

The control of a plant with a controller whose form is specified beforehand will be defined as specific control and the problem of choosing the parameters in the specific controller to extremize a performance index will be defined as the optimal specific control problem.

It will be assumed here that the decision problem of adaptive control systems is equivalent to the optimal specific control problem.

The optimal specific control problem will be stated in the following manner.

Given a plant with dynamic Eq. (14) and initial condition (15) determine the unknown parameters in a control law of the form

$$u = l(\underline{y}, \underline{b}) \quad (25)$$

where  $\underline{y}$  is a  $p$ -dimensional subset of  $\underline{x}$  and  $\underline{b}$  is a  $q$ -dimensional parameter vector such that a performance index of the form (16) is minimized.

The following methods are proposed for solving the optimization problem.

#### A. THE QUASILINEARIZATION METHOD FOR SPECIFIC CONTROL

The optimal specific control problem may be rewritten as:

Given

$$\dot{\underline{x}} = \underline{h}(\underline{x}, \underline{b}) \quad (26)$$

with  $\underline{x}(0) = \underline{c}$  where

$$\underline{h}(\underline{x}, \underline{b}) = \underline{f}(\underline{x}, l(\underline{y}, \underline{b})) , \quad (27)$$

determine the parameter vector  $\underline{b}$  such that

$$J(\underline{b}) = \int_0^T k(\underline{x}, \underline{b}) dt \quad (28)$$

is minimized.

In Eq. (28)

$$k(\underline{x}, \underline{b}) = g(\underline{x}, l(\underline{y}, \underline{b})) \quad (29)$$

By adjoining the equation

$$\dot{\underline{b}} = 0 \quad (30)$$

to Eq. (26) the specific control problem is reduced to an ordinary problem of minimization of an integral with differential constraints. The Euler equations for the minimization problem are, in addition to Eqs. (26) and (30)

$$\dot{\underline{\lambda}} = - \underline{k}_{\underline{x}} - \underline{h}'_{\underline{x}} \underline{\lambda} \quad (31)$$

$$\dot{\underline{\mu}} = - \underline{k}_{\underline{b}} - \underline{h}'_{\underline{b}} \underline{\mu} \quad (32)$$

where  $\underline{\lambda}$  and  $\underline{\mu}$  are the Lagrange multipliers associated with Eqs. (26) and (30) respectively. The quantities  $\underline{k}_{\underline{x}}$ ,  $\underline{k}_{\underline{b}}$ ,  $\underline{h}'_{\underline{x}}$  and  $\underline{h}'_{\underline{b}}$  are defined in a manner analogous to Eqs. (19) and (20).

The natural boundary conditions for the problem are

$$\underline{\lambda}(T) = 0 \quad (33)$$

$$\underline{\mu}(0) = \underline{\mu}(T) = 0 \quad (34)$$

The Euler equations with the boundary conditions may now be solved by the method of quasilinearization to yield the values of the parameters in the optimal specific controller.

### B. OPEN LOOP - CLOSED LOOP METHOD FOR DECISION

Philosophically, this method is different from the previous method. The solution here depends on the solutions given by Eqs. (23) and (24) which do not depend on the form of the specific controller. In general, the solution using this method will result in a slight degradation of performance compared to the previous method. However, this method will yield solutions which will be easier to instrument. The computations necessary with this method are often easier to perform.

Recalling that Eq. (24) represents the optimal trajectory without the specific controller constraint and Eq. (26) represents the system equation with a specific controller, it is easy to see that if there exists a set  $\underline{b}$  such that

$$\dot{\underline{\phi}} = \underline{h}(\underline{\phi}, \underline{b}) \quad (35)$$

then the set is the optimal parameter values for the specific controller. However, in general, Eq. (35) cannot be satisfied.

Equation (35) may be rewritten as

$$\dot{\underline{\phi}} - \underline{h}(\underline{\phi}, \underline{b}) \equiv 0 \quad (36)$$

Since Eq. (36) cannot be satisfied in general, one intuitively feels that an acceptable solution may be one which makes the left-hand side of (36) "close to zero," the closeness being defined in a suitable manner. In this Memorandum, the set  $\underline{b}$  will be picked such that

$$\int_0^T < \dot{\underline{\varphi}} - \underline{h}(\underline{\varphi}, \underline{b}) , \quad \dot{\underline{\varphi}} - \underline{h}(\underline{\varphi}, \underline{b}) > dt \quad (37)$$

is minimized with respect to  $\underline{b}$ .

The method of solving this minimization problem is straightforward and is exactly similar to the method used in the differential-approximation method for identification. The partial derivatives of Eq. (37) are taken with respect to the components of  $\underline{b}$  and equated to zero. This will yield  $q$  simultaneous equations involving the  $q$  component of  $\underline{b}$ . The solution of this set of equations yields the optimal specific controller.

A useful alternative method is proposed by Strejc.<sup>(12)</sup>

#### IV. DISCUSSION

Once the controller parameters are determined, the modification problem consists of resetting the parameters to the determined values. This is the easiest part of the over-all problem of designing an adaptive controller.

Unified methods for solving the identification and decision problems of adaptive control are proposed in this Memorandum. Use of these methods will permit the same equipment to perform both operations. This will result in cheaper cost for adaptive controllers.

To utilize the methods here, the adaptive controller can be either digital, analog or a combination of both.

# REFERENCES

1. Zadeh, L. A., "On the Definition of Adaptivity," Proc. of the IEEE, Vol. 51, No. 3, March 1963, pp. 469-470.
2. Gibson, J. E., K. S. Fu, J. B. Pearson, Z. V. Rekosius, and R. Sridhar, Modern Aspects of Automatic Control, Vol. 2, Chap. 7, Purdue University, June 1963.
3. Kalaba, R. E., "Some Aspects of Quasilinearization," a chapter in the book Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York, 1963.
4. Bellman, R. E., R. E. Kalaba, and B. Kotkin, Differential Approximation Applied to the Solution of Convolution Equations, The RAND Corporation, RM-3601, May 1963.
5. Gibson, J. E., Nonlinear Automatic Control, Chapt. 11, McGraw-Hill, 1963.
6. Bellman, R. E., and R. E. Kalaba, "Dynamic Programming and Adaptive Processes: Mathematical Foundation," IRE Trans. on Autom. Control, Vol. AC-5 (1960), pp. 5-10.
7. Bellman, R. E., R. E. Kalaba, and H. Kagiwada, Quasilinearization System Identification and Prediction, The RAND Corporation, RM-3812, August 1963.
8. Bellman, R. E., and R. E. Kalaba, Quasilinearization and Nonlinear Boundary Value Problems, American Elsevier Publishing Co., New York, to appear in 1964.
9. Bellman, R. E., R. E. Kalaba, and H. Kagiwada, "Orbit Determination as a Multi-point Boundary Value Problem and Quasilinearization," Proc. Nat. Acad. Sci. USA, Vol. 48 (1962), pp. 1327-1329.
10. Kalaba, R. E., "On Nonlinear Differential Equations, the Maximum Operation and Monotone Convergence," J. of Math. and Mech., Vol. 8 (1959), pp. 519-574.
11. Berkovitz, L. D., "Variational Methods in Problems of Control and Programming," J. Math. Anal. and App., Vol. 3, No. 1, August 1961.
12. Strejc, V., "Evaluation of General Signals with Non-zero Initial Conditions," Acta Techn., Vol. 6 (1961), pp. 378-391.